

January 16, 1896.

Sir JOSEPH LISTER, Bart., President, in the Chair.

A List of the Presents received was laid on the table, and thanks ordered for them.

The following Papers were read:—

- I. "The Rotation of an Elastic Spheroid." By S. S. HOUGH, B.A., Fellow of St. John's College, and Isaac Newton Student in the University of Cambridge. Communicated by Professor G. H. DARWIN, F.R.S. Received December 12, 1895.

(Abstract.)

1. It is well known that if a rigid body, whose principal moments of inertia are A, A, C , be set rotating about its axis of symmetry and then be subjected to a slight disturbance, it will execute oscillations about its mean position, in consequence of which the axis of rotation will undergo periodic displacements relatively to the body in a period which bears to the period of rotation the ratio $A : C - A$. The object of the present investigation is to determine to what extent this period will be modified if the body, instead of being perfectly rigid, is capable of elastic deformations.

The analysis is confined to the case of a homogeneous spheroid of revolution composed of isotropic, incompressible, gravitating material, while for further simplicity the body is supposed to be free from strain in its interior when rotating uniformly. If we refer to a set of rectangular axes rotating with uniform angular velocity ω about the axis of z , the rigorous differential equations for the vibrations of such a body are

$$\left. \begin{aligned} n\nabla^2 u + \frac{\partial \psi}{\partial x} &= \ddot{u} - 2\omega \dot{v} \\ n\nabla^2 v + \frac{\partial \psi}{\partial y} &= \ddot{v} + 2\omega \dot{u} \\ n\nabla^2 w + \frac{\partial \psi}{\partial z} &= \ddot{w} \\ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} &= 0 \end{aligned} \right\} \dots\dots\dots (1),$$

where u, v, w denote the displacements at the point x, y, z ; ψ is a function depending on pressure and potential at this point, and $n = \kappa/\rho$, where κ is the rigidity and ρ the density.

Putting u, v, w, ψ each proportional to $e^{i\lambda t}$, and performing the differentiations with regard to the time, the above equations become

$$\left. \begin{aligned} (n\nabla^2 + \lambda^2)u + 2\omega i\lambda v &= -\frac{\partial\psi}{\partial x} \\ (n\nabla^2 + \lambda^2)v - 2\omega i\lambda u &= -\frac{\partial\psi}{\partial y} \\ (n\nabla^2 + \lambda^2)w &= -\frac{\partial\psi}{\partial z} \\ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} &= 0 \end{aligned} \right\} \dots\dots\dots (2).$$

These equations are theoretically sufficient to determine u, v, w, ψ , subject to certain boundary conditions. The boundary conditions should also lead to a series of values of λ , which correspond to the different periods of free oscillation of the system.

2. The boundary conditions express the fact that the free surface is not subject to stress. Denoting by $\cos \alpha, \cos \beta, \cos \gamma$ the direction cosines of the normal to the mean surface, from this condition we obtain the following equations, which hold good at the undisturbed surface:—

$$\left. \begin{aligned} \psi \cos \alpha + n \left\{ \frac{du}{dx} \cos \alpha + \frac{dv}{dy} \cos \beta + \frac{dw}{dz} \cos \gamma + \frac{dw}{dx} \cos \alpha + \frac{dv}{dx} \cos \beta \right. \\ \left. + \frac{dw}{dx} \cos \gamma \right\} &= (v' - g\zeta) \cos \alpha \\ \psi \cos \beta + n \left\{ \frac{dv}{dx} \cos \alpha + \frac{dv}{dy} \cos \beta + \frac{dv}{dz} \cos \gamma + \frac{du}{dy} \cos \alpha + \frac{dv}{dy} \cos \beta \right. \\ \left. + \frac{dw}{dy} \cos \gamma \right\} &= (v' - g\zeta) \cos \beta \\ \psi \cos \gamma + n \left\{ \frac{dw}{dx} \cos \alpha + \frac{dw}{dy} \cos \beta + \frac{dw}{dz} \cos \gamma + \frac{du}{dz} \cos \alpha + \frac{dv}{dz} \cos \beta \right. \\ \left. + \frac{dw}{dz} \cos \gamma \right\} &= (v' - g\zeta) \cos \gamma \end{aligned} \right\} (3),$$

where g denotes the value of gravity at the surface, ζ the height of the surface waves, and v' the potential due to the harmonic inequalities.

3. To obtain a solution of the equations of motion subject to these boundary conditions, suppose the displacements u, v, w consist of two parts: 1st, u_0, v_0, w_0 , due to the rotation of the body as a whole through small angles θ_1, θ_2 about Ox, Oy ; and 2nd, u_1, v_1, w_1 , due to elastic distortion. Then—

$$\begin{aligned} u &= u_0 + u_1 = z\theta_2 + u_1, \\ v &= v_0 + v_1 = -z\theta_1 + v_1, \\ w &= w_0 + w_1 = x\theta_2 - y\theta_1 + w_1. \end{aligned}$$

Changing the variables from u, v, w to u_1, v_1, w_1 the equations of motion take the form

$$\left. \begin{aligned} (n\nabla^2 + \lambda^2)u_1 + 2\omega i\lambda v_1 &= -\frac{\partial \psi}{\partial x} - \lambda^2 z\theta_2 + 2\omega i\lambda z\theta_1 \\ (n\nabla^2 + \lambda^2)v_1 - 2\omega i\lambda u_1 &= -\frac{\partial \psi}{\partial y} + \lambda^2 z\theta_1 + 2\omega i\lambda z\theta_2 \\ (n\nabla^2 + \lambda^2)w_1 &= -\frac{\partial \psi}{\partial z} + \lambda^2(x\theta_2 - y\theta_1) \\ \frac{du_1}{dx} + \frac{dv_1}{dy} + \frac{dw_1}{dz} &= 0 \end{aligned} \right\} \dots (4),$$

while the boundary conditions become

$$\left. \begin{aligned} \psi \cos \alpha + n \left\{ \frac{du_1}{dx} \cos \alpha + \frac{dv_1}{dy} \cos \beta + \frac{dw_1}{dz} \cos \gamma + \frac{du_1}{dx} \cos \alpha + \frac{dv_1}{dx} \cos \beta \right. \\ \left. + \frac{dw_1}{dx} \cos \gamma \right\} - (v'_1 - g\zeta_1) \cos \alpha &= \omega^2(\theta_2 xz - \theta_1 yz) \cos \alpha \\ \psi \cos \beta + n \left\{ \frac{dv_1}{dx} \cos \alpha + \frac{dv_1}{dy} \cos \beta + \frac{dw_1}{dz} \cos \gamma + \frac{du_1}{dy} \cos \alpha + \frac{dv_1}{dy} \cos \beta \right. \\ \left. + \frac{dw_1}{dy} \cos \gamma \right\} - (v'_1 - g\zeta_1) \cos \beta &= \omega^2(\theta_2 xz - \theta_1 yz) \cos \beta \\ \psi \cos \gamma + n \left\{ \frac{dw_1}{dx} \cos \alpha + \frac{dw_1}{dy} \cos \beta + \frac{dw_1}{dz} \cos \gamma + \frac{du_1}{dz} \cos \alpha + \frac{dv_1}{dz} \cos \beta \right. \\ \left. + \frac{dw_1}{dz} \cos \gamma \right\} - (v'_1 - g\zeta_1) \cos \gamma &= \omega^2(\theta_2 xz - \theta_1 yz) \cos \gamma \end{aligned} \right\} (5),$$

where ζ_1 denotes the height of the surface waves consequent on the displacements u_1, v_1, w_1 , and v'_1 the part of v' due to the harmonic inequalities ζ_1 . The advantage of this transformation consists in the

fact that in the oscillation with which we are concerned u_1, v_1, w_1, ψ will all be small quantities of the order of the ellipticity of the spheroid, while λ/ω will be small of the order ω^3 or $\epsilon^{3/2}$. Hence, if we neglect small quantities of the order of the square of the ellipticity the equations (4) reduce to

$$\left. \begin{aligned} n\nabla^2 u_1 &= -\frac{\partial \psi}{\partial x}, & n\nabla^2 v_1 &= -\frac{\partial \psi}{\partial y}, & n\nabla^2 w_1 &= -\frac{\partial \psi}{\partial z} \\ \frac{du_1}{dx} + \frac{dv_1}{dy} + \frac{dw_1}{dz} &= 0 \end{aligned} \right\} \dots (6),$$

while to the same order of approximation the boundary equations (5) may be supposed to hold good at the surface of a sphere $r = a$.

4. The equations (6), with the boundary conditions (5) at the surface of the sphere $r = a$, are the equations obtained in considering the equilibrium of a sphere distorted by forces throughout its mass derivable from a potential function $\omega^2(\theta_2 xz - \theta_1 yz)$. By using the well-known solution of this problem we therefore obtain

$$\left. \begin{aligned} u &= u_0 + u_1 = z\theta_2 \left\{ 1 + \left(\frac{8}{5} - \frac{r^2}{a^2} \right) \frac{\epsilon'}{1 + \epsilon'/\epsilon} \right\} + \frac{4}{5} \frac{\epsilon'}{1 + \epsilon'/\epsilon} \frac{x(\theta_2 xz - \theta_1 yz)}{a^2} \\ v &= v_0 + v_1 = z\theta_1 \left\{ 1 + \left(\frac{8}{5} - \frac{r^2}{a^2} \right) \frac{\epsilon'}{1 + \epsilon'/\epsilon} \right\} + \frac{4}{5} \frac{\epsilon'}{1 + \epsilon'/\epsilon} \frac{y(\theta_2 xz - \theta_1 yz)}{a^2} \\ w &= w_0 + w_1 = (-x\theta_2 + y\theta_1) \left\{ 1 - \left(\frac{8}{5} - \frac{r^2}{a^2} \right) \frac{\epsilon'}{1 + \epsilon'/\epsilon} \right\} \\ &\quad + \frac{4}{5} \frac{\epsilon'}{1 + \epsilon'/\epsilon} \frac{z(\theta_2 xz - \theta_1 yz)}{a^2} \end{aligned} \right\} (7),$$

$$\text{where } \epsilon = \frac{15\omega^2}{16\pi\rho}, \quad \epsilon' = \frac{5\omega^2 a^2}{38n}.$$

We have thus expressed u, v, w by means of two arbitrary constants θ_1, θ_2 . If with these values of u, v, w we form expressions for the angular momenta of the body about the axes Ox, Oy , and then express that the rates of change of these angular momenta are zero, we obtain two equations giving the ratios of the quantities θ_1, θ_2 and involving λ . Eliminating θ_1, θ_2 from these equations we are led to an equation for λ , which is found to reduce to

$$\lambda/\omega = \frac{\epsilon}{1 + \epsilon'/\epsilon}.$$

5. If we had neglected the elastic distortions we should have obtained to the same order of approximation

$$\lambda/\omega = \epsilon.$$

Hence, the effect of elastic distortion is to prolong the period in the ratio $1 + \epsilon'/\epsilon : 1$. Taking a spheroid of the same size and mean density as the Earth rotating in a sidereal day, we find that the period is extended from 232 days to 335 days if the rigidity be that of steel. Assuming that in the case of the actual Earth the effects of heterogeneity are to still further prolong the period in the ratio 305 : 232, we find for the period of oscillation of the Earth, supposed of the rigidity of steel, 440 days.

Now, observation* indicates that the earth is undergoing such an oscillation as we have been discussing in a period of 427 days instead of the period of 305 days indicated by theory on the assumption that the motion would take place sensibly in the same manner as if the earth were rigid. We see now that the divergence between theory and observation can be explained by taking into account the elastic distortions of the solid parts of the earth, and that the degree of rigidity required to account for the observed period is slightly in excess of the rigidity of steel.

This explanation has been previously offered by Professor Newcomb,† who has treated the problem by a simple geometrical method. The main object in undertaking the present analytical investigation was to examine the validity of certain hypotheses made by Newcomb. An examination of the type of oscillation indicated by our analysis shows that the procedure adopted by Newcomb is legitimate, but that there is a slight error in his assumed law of displacement of the pole of figure due to centrifugal force. Our method has the additional advantage of showing the degree of accuracy to which the results may be expected to hold good.

II. "On a Type of Spherical Harmonics of unrestricted Degree, Order, and Argument." By E. W. HOBSON, Sc.D., F.R.S. Received December 23, 1895.

(Abstract.)

The type of harmonics considered in this memoir is

$$r^n \frac{\cos}{\sin} m\phi \cdot u_n^m(\mu),$$

where $u_n^m(\mu)$ satisfies the differential equation

$$(1 - \mu^2) \frac{d^2 u}{d\mu^2} - 2\mu \frac{du}{d\mu} + \left\{ n(n+1) - \frac{m^2}{1 - \mu^2} \right\} u = 0,$$

* *Vide* Chandler, 'Astron. Journal,' vols. 11, *et seq.*

† 'Monthly Notices of the Royal Astron. Soc.,' March, 1892.